

# BASINS OF ATTRACTION IN LOEWNER EQUATIONS

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**ABSTRACT.** We prove that any Loewner PDE whose driving term  $h(z, t)$  vanishes at the origin, and satisfies the bunching condition  $\ell m(Dh(0, t)) \geq k(Dh(0, t))$  for some  $\ell \in \mathbb{R}^+$ , admits a solution given by univalent mappings  $(f_t: \mathbb{B}^q \rightarrow \mathbb{C}^q)_{t \geq 0}$ . This is done by discretizing time and considering the abstract basin of attraction. If  $\ell < 2$ , then the range  $\cup_{t \geq 0} f_t(\mathbb{B}^q)$  of any such solution is biholomorphic to  $\mathbb{C}^q$ .

## 1. INTRODUCTION

Let  $\mathbb{B}^q \subset \mathbb{C}^q$  denote the unit ball. The Loewner PDE

$$\frac{\partial f_t(z)}{\partial t} = Df_t(z)h(z, t), \quad \text{a.e. } t \geq 0, \quad z \in \mathbb{B}^q \quad (1.1)$$

was introduced by Loewner [21] and developed by Kufarev [20] and Pommerenke [26] in the case of the unit disc  $\mathbb{D} \doteq \mathbb{B}^1$ . The study of this equation culminated with the proof of the Bieberbach conjecture by de Branges [9] and the introduction of the stochastic Loewner evolution by Schramm [27].

The several variables case has been widely studied for its application in geometric function theory by Graham, Hamada, G.Kohr, M.Kohr, Pfaltzgraff and others (see e.g. [17] [25]).

In [5], generalizing the results obtained in the unit disc  $\mathbb{D}$  in [8], we explore the connections between this topic and the theory recently developed by Bracci, Contreras and Díaz-Madrigal [6] [7] (see also [4]) of Herglotz non-autonomous vector fields on complete hyperbolic manifolds. An *Herglotz vector field of order  $\infty$*  on  $\mathbb{B}^q$  is a non-autonomous holomorphic vector field  $-h(z, t): \mathbb{B}^q \times \mathbb{R}^+ \rightarrow \mathbb{C}^q$  such that

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- $-h(z, t)$  is measurable in  $t \geq 0$  and for a.e.  $\bar{t} \geq 0$ , the holomorphic vector field  $-h(z, \bar{t})$  is an *infinitesimal generator*, that is the “frozen” Cauchy problem

$$\begin{cases} \dot{z}(s) = -h(z(s), \bar{t}), \\ z(0) = z_0, \end{cases}$$

has a solution  $z : [0, +\infty) \rightarrow \mathbb{B}^q$  for all  $z_0 \in \mathbb{B}^q$ ,

- for any compact set  $K \subset \mathbb{B}^q$  and any  $T > 0$  there exists  $c_{K,T} > 0$  satisfying

$$|h(z, t)| \leq c_{K,T}, \quad z \in K, 0 \leq t \leq T.$$

The solution flow of the Loewner ODE

$$\begin{cases} \frac{\partial}{\partial t} \varphi_{s,t}(z) = -h(\varphi_{s,t}(z), t), & z \in \mathbb{B}^q, \text{ a.e. } t \in [s, \infty), \\ \varphi_{s,s}(z) = z, & z \in \mathbb{B}^q, s \geq 0, \end{cases} \quad (1.2)$$

is an *evolution family of order  $\infty$* , that is a family of holomorphic mappings  $(\varphi_{s,t} : \mathbb{B}^q \rightarrow \mathbb{B}^q)_{0 \leq s \leq t}$  satisfying

- $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$  for all  $0 \leq s \leq u \leq t$  and  $\varphi_{s,s}(z) = z$  for all  $s \geq 0$ ,
- for any compact set  $K \subset \mathbb{B}^q$  and for any  $T > 0$  there exists a  $C_{K,T} > 0$  satisfying

$$|\varphi_{s,t}(z) - \varphi_{s,u}(z)| \leq C_{K,T}(t - u), \quad z \in K, 0 \leq s \leq u \leq t < T. \quad (1.3)$$

In [5] we prove that a family  $(f_t : \mathbb{B}^q \rightarrow \mathbb{C}^q)_{t \geq 0}$  of univalent mappings is locally Lipschitz (in the variable  $t$ ) and solves the Loewner PDE (1.1) if and only if it solves the functional equation

$$f_s = f_t \circ \varphi_{s,t}, \quad 0 \leq s \leq t. \quad (1.4)$$

If such a solution  $(f_t : \mathbb{B}^q \rightarrow \mathbb{C}^q)$  exists, then the subset  $\bigcup_{t \geq 0} f_t(\mathbb{B}^q) \subset \mathbb{C}^q$  is open and connected and is called the *range* of  $(f_t)$ . Any other solution  $(g_t : \mathbb{B}^q \rightarrow \mathbb{C}^q)$  is of the form  $(\Lambda \circ f_t)$ , where  $\Lambda : \bigcup_{t \geq 0} f_t(\mathbb{B}^q) \rightarrow \mathbb{C}^q$  is holomorphic. Thus the ranges of two univalent solutions of (1.1) are biholomorphic.

We are interested in Herglotz vector fields on  $\mathbb{B}^q$  whose flow  $(\varphi_{s,t})$  is attracting in the origin. A first example is provided by Herglotz vector fields whose linear part does not depend on  $t \geq 0$ . This has been studied in [10][15].

**Theorem 1.1.** *Let  $-h(z, t)$  be a Herglotz vector field of order  $\infty$  on  $\mathbb{B}^q$  such that  $h(z, t) = Az + O(|z|^2)$  with*

$$2 \min\{\operatorname{Re} \langle Az, z \rangle : |z| = 1\} > \max\{\operatorname{Re} \lambda : \lambda \in \operatorname{sp}(A)\}, \quad (1.5)$$

*where  $\langle \cdot, \cdot \rangle$  is the hermitian product on  $\mathbb{C}^q$ . Then the Loewner PDE (1.1) admits a locally Lipschitz univalent solution  $(f_t : \mathbb{B}^q \rightarrow \mathbb{C}^q)$ . The range  $\bigcup_{t \geq 0} f_t(\mathbb{B}^q)$  of any such solution is biholomorphic to  $\mathbb{C}^q$ .*

This result was generalized in [3] (see also [2]), with an approach based on a discretization of time.

**Theorem 1.2.** *Let  $-h(z, t)$  be a Herglotz vector field of order  $\infty$  on  $\mathbb{B}^q$  such that  $h(z, t) = Az + O(|z|^2)$ , where the eigenvalues of  $A$  have strictly positive real part. Then the Loewner PDE (1.1) admits a locally Lipschitz univalent solution  $(f_t: \mathbb{B}^q \rightarrow \mathbb{C}^q)$ . The range  $\bigcup_{t \geq 0} f_t(\mathbb{B}^q)$  of any such solution is biholomorphic to  $\mathbb{C}^q$ .*

The same result was obtained independently with different methods by Voda [28], assuming  $\min\{\operatorname{Re} \langle Az, z \rangle : |z| = 1\} > 0$ . See also [12] for related results.

The next natural step is admitting time-dependent linear parts. Set

$$m(A) \doteq \min\{\operatorname{Re} \langle Az, z \rangle : |z| = 1\}, \quad k(A) = \max\{\operatorname{Re} \langle Az, z \rangle : |z| = 1\}.$$

The following result is proved in [16][14]:

**Theorem 1.3.** *Let  $-h(z, t)$  be a Herglotz vector field on  $\mathbb{B}^q$  of order  $\infty$  such that  $h(z, t) = A(t)z + O(|z|^2)$ , and assume that the family of linear mappings  $(A(t))_{t \geq 0}$  satisfies:*

- i)  $m(A(t)) > 0$  for all  $t \geq 0$  and  $\int_0^\infty m(A(t)) = \infty$ ,
- ii)  $t \mapsto \|A(t)\|$  is uniformly bounded on  $\mathbb{R}^+$ ,
- iii) there exists  $\delta > 0$  such that

$$2m(A(t)) \geq k(A(t)) + \delta, \quad t \geq 0,$$

iv)

$$\int_s^t A(\tau) d\tau \circ \int_r^s A(\tau) d\tau = \int_r^s A(\tau) d\tau \circ \int_s^t A(\tau) d\tau, \quad t \geq s \geq r \geq 0.$$

*Then the Loewner PDE (1.1) admits a locally Lipschitz univalent solution  $(f_t: \mathbb{B}^q \rightarrow \mathbb{C}^q)$ . The range  $\bigcup_{t \geq 0} f_t(\mathbb{B}^q)$  of any such solution is biholomorphic to  $\mathbb{C}^q$ .*

In this paper we generalize Theorem 1.3, using the approach of [2][3]. The following is our result.

**Theorem 1.4.** *Let  $-h(z, t)$  be a Herglotz vector field on  $\mathbb{B}^q$  of order  $\infty$  such that  $h(z, t) = A(t)z + O(|z|^2)$ , and assume that the family of linear mappings  $(A(t))_{t \geq 0}$  satisfies:*

- a)  $m(A(t)) > 0$  for all  $t \geq 0$  and  $\int_0^\infty m(A(t)) = \infty$ ,
- b)  $t \mapsto \|A(t)\|$  is locally bounded on  $\mathbb{R}^+$ ,
- c) there exists  $\ell \in \mathbb{R}^+$  such that

$$\ell m(A(t)) \geq k(A(t)), \quad t \geq 0.$$

*Then the Loewner PDE (1.1) admits a locally Lipschitz univalent solution  $(f_t: \mathbb{B}^q \rightarrow \mathbb{C}^q)$ . If  $\ell < 2$  then the range  $\bigcup_{t \geq 0} f_t(\mathbb{B}^q)$  of any such solution is biholomorphic to  $\mathbb{C}^q$ .*

Notice that the assumptions ii) and iii) of Theorem 1.3 imply that there exists  $\ell < 2$  such that

$$\ell m(A(t)) \geq k(A(t)), \quad t \geq 0.$$

We want to stress the strong analogy between Loewner theory and the theory of discrete non-autonomous complex dynamical systems grown around Bedford's conjecture

(see [1][11][19][23][30]). This is reflected in the proof of Theorem 1.4, which is based on a discretization of time, and relies on the study of the abstract basin of attraction performed by Fornæss and Stensønes in [11]:

**Theorem 1.5.** *Let  $(\varphi_{n,n+1})_{n \in \mathbb{N}}$  be a family of univalent self-mappings of  $r\mathbb{B}^q$ . Assume that there exist  $0 < \nu \leq \mu < 1$  such that*

$$\nu|z| \leq |\varphi_{n,n+1}(z)| \leq \mu|z|, \quad z \in r\mathbb{B}^q, n \in \mathbb{N}. \quad (1.6)$$

*Then, if  $\Omega$  is the abstract basin of attraction of  $(\varphi_{n,n+1})$ , then there exists an univalent mapping  $\Psi: \Omega \rightarrow \mathbb{C}^q$ .*

The abstract basin of attraction comes naturally with a family of univalent mappings  $(\omega_n: r\mathbb{B}^q \rightarrow \Omega)$ . Composing this family with the biholomorphism  $\Psi$  given by Theorem 1.5 we obtain a family of univalent mappings  $r\mathbb{B}$  to  $\mathbb{C}^q$  which we extend to a family  $(f_t: \mathbb{B}^q \rightarrow \mathbb{C}^q)$  satisfying the functional equation (1.4).

The range  $\bigcup_{t \geq 0} f_t(\mathbb{B}^q)$  is by construction biholomorphic to  $\Omega$  and thus by [11, Theorem 3.1] it is a Stein, Runge domain in  $\mathbb{C}^q$  whose Kobayashi pseudometric vanishes identically and which is diffeomorphic to  $\mathbb{C}^q$ . It is an open question whether  $\bigcup_{t \geq 0} f_t(\mathbb{B}^q)$  is biholomorphic to  $\mathbb{C}^q$  when  $\ell \geq 2$ . A positive answer would follow from a proof of Bedford's conjecture (see e.g. [22]):

**Conjecture 1.6.** *Let  $(\Phi_{n,n+1})_{n \in \mathbb{N}}$  be a family of automorphisms of  $\mathbb{C}^q$ . Assume that there exist  $0 < \nu \leq \mu < 1$  and  $r > 0$  such that*

$$\nu|z| \leq |\Phi_{n,n+1}(z)| \leq \mu|z|, \quad z \in r\mathbb{B}^q, n \in \mathbb{N}. \quad (1.7)$$

*Then the basin of attraction*

$$\{z \in \mathbb{C}^q : \lim_{n \rightarrow \infty} \Phi_{n-1,n} \circ \cdots \circ \Phi_{0,1} = 0\}$$

*is biholomorphic to  $\mathbb{C}^q$ .*

## 2. MAIN RESULT

Let  $\mathcal{N}$  denote the family of holomorphic mappings  $h: \mathbb{B}^q \rightarrow \mathbb{C}^q$  such that  $h(0) = 0$  and  $\operatorname{Re} \langle h(z), z \rangle > 0$ , for all  $z \neq 0$ .

**Theorem 2.1.** *Let  $h(z, t): \mathbb{B}^q \times \mathbb{R}^+ \rightarrow \mathbb{C}^q$  be a mapping such that  $z \mapsto h(z, t) \in \mathcal{N}$  for all  $t \in \mathbb{R}^+$  and  $t \mapsto h(z, t)$  is measurable on  $\mathbb{R}^+$  for all  $z \in \mathbb{B}^q$ . Assume that  $h(z, t) = A(t)z + O(|z|^2)$  and that the family of linear mappings  $(A(t))_{t \geq 0}$  satisfies:*

- a)  $m(A(t)) > 0$  for all  $t \geq 0$  and  $\int_0^\infty m(A(t)) = \infty$ ,
- b)  $t \mapsto \|A(t)\|$  is locally bounded on  $\mathbb{R}^+$ ,
- c) there exists  $\ell \in \mathbb{R}^+$  such that

$$\ell m(A(t)) \geq k(A(t)), \quad t \geq 0.$$

Then the Loewner PDE

$$\frac{\partial f_t(z)}{\partial t} = Df_t(z)h(z, t), \quad z \in \mathbb{B}^q, \text{ a.e. } t \geq 0$$

admits a global solution given by univalent mappings  $(f_t: \mathbb{B}^q \rightarrow \mathbb{C}^q)$ . If  $l < 2$ , then the range  $\bigcup_{t \geq 0} f_t(\mathbb{B}^q)$  of any such solution is biholomorphic to  $\mathbb{C}^q$ . Any other solution given by holomorphic mappings  $(g_t: \mathbb{B}^q \rightarrow \mathbb{C}^q)$  is of the form  $(\Lambda \circ f_t)$ , where  $\Lambda: \bigcup_{t \geq 0} f_t(\mathbb{B}^q) \rightarrow \mathbb{C}^q$  is holomorphic.

*Proof.* Notice that for all  $A \in \mathcal{L}(\mathbb{C}^q)$ ,

$$m(A) \leq k(A) \leq \|A\|,$$

and thus  $k(t)$  and  $m(t)$  are also locally bounded on  $\mathbb{R}^+$ . By [15, Lemma 1.2] one has for a.e.  $t \geq 0$

$$|h(z, t)| \leq \frac{4r}{(1-r)^2} \|A(t)\|, \quad |z| \leq r < 1,$$

hence  $-h(z, t)$  is a Herglotz vector field of order  $\infty$  on  $\mathbb{B}^q$ . Let  $(\varphi_{s,t})$  be the associated evolution family of order  $\infty$ , that is the solution of the Loewner ODE

$$\begin{cases} \frac{\partial}{\partial t} \varphi_{s,t}(z) = -h(\varphi_{s,t}(z), t), & z \in \mathbb{B}^q, \text{ a.e. } t \in [s, \infty), \\ \varphi_{s,s}(z) = z, & z \in \mathbb{B}^q, s \geq 0. \end{cases} \quad (2.1)$$

Recall that  $\varphi_{s,t}: \mathbb{B}^q \rightarrow \mathbb{B}^q$  is an univalent mapping for all  $0 \leq s \leq t$  and that  $t \mapsto \varphi_{s,t}(z)$  is locally Lipschitz continuous on  $[s, \infty)$  uniformly on compact sets with respect to  $z \in \mathbb{B}^q$ .

Fix  $s \geq 0$  and  $z \in \mathbb{B}^q \setminus \{0\}$ . Then for a.e.  $\tau \geq 0$ ,

$$\frac{\partial}{\partial \tau} |\varphi_{s,\tau}(z)|^2 = 2 \operatorname{Re} \left\langle \frac{\partial}{\partial \tau} \varphi_{s,\tau}(z), \varphi_{s,\tau}(z) \right\rangle = -2 \operatorname{Re} \langle h(\varphi_{s,\tau}(z), \tau), \varphi_{s,\tau}(z) \rangle.$$

Set  $C(r) \doteq \frac{1+r}{1-r}$  and  $c(r) \doteq \frac{1-r}{1+r}$  for all  $r \geq 0$ . Gurganus proved [18] that for a.e.  $t \geq 0$ ,

$$\operatorname{Re} \langle A(t)w, w \rangle c(|w|) \leq \operatorname{Re} \langle h(w, t), w \rangle \leq \operatorname{Re} \langle A(t)w, w \rangle C(|w|), \quad w \in \mathbb{B}^q \setminus \{0\}.$$

Since  $|\varphi_{s,\tau}(z)| \leq |z|$  one has using assumption d),

$$\begin{aligned} -2k(A(\tau))C(|z|) &\leq \frac{\frac{\partial}{\partial \tau} |\varphi_{s,\tau}(z)|^2}{|\varphi_{s,\tau}(z)|^2} \leq -2m(A(\tau))c(|z|), \quad \text{a.e. } \tau \geq 0, \\ -2C(|z|) \int_s^t k(A(\tau)) d\tau &\leq \int_s^t \frac{\frac{\partial}{\partial \tau} |\varphi_{s,\tau}(z)|^2}{|\varphi_{s,\tau}(z)|^2} d\tau \leq -2c(|z|) \int_s^t m(A(\tau)) d\tau, \quad 0 \leq s \leq t, \\ e^{-C(|z|) \int_s^t k(A(\tau)) d\tau} &\leq \frac{|\varphi_{s,t}(z)|}{|z|} \leq e^{-c(|z|) \int_s^t m(A(\tau)) d\tau}, \quad 0 \leq s \leq t. \end{aligned} \quad (2.2)$$

Set for all  $0 \leq s \leq t$ ,

$$\nu_{s,t} \doteq e^{-C(|z|) \int_s^t k(A(\tau)) d\tau}, \quad \text{and} \quad \mu_{s,t} \doteq e^{-c(|z|) \int_s^t m(A(\tau)) d\tau}.$$

One has, thanks to assumption c),

$$\log_{\mu_{s,t}} \nu_{s,t} = \frac{\log \nu_{s,t}}{\log \mu_{s,t}} = C^2(|z|) \frac{\int_s^t k(A(\tau)) d\tau}{\int_s^t m(A(\tau)) d\tau} \leq C^2(|z|) \ell, \quad 0 \leq s \leq t. \quad (2.3)$$

Let  $n \in \mathbb{N}$  and let  $u_n \in \mathbb{R}^+$  be defined by

$$\int_0^{u_n} m(A(\tau)) d\tau = n.$$

Let now  $h \in \mathbb{N}$  be the least integer strictly greater than  $\ell$ , and let  $r > 0$  be such that  $C^2(r) < h/\ell$ . Set  $\mu \doteq e^{-c(r)}$  (notice that  $\mu = \mu_{u_n, u_{n+1}}$  for all  $n \geq 0$ ) and  $\nu \doteq \min\{\nu_{u_n, u_{n+1}} : n \geq 0\}$ . By (2.2) and (2.3) one has that

$$\nu|z| \leq |\varphi_{u_n, u_{n+1}}(z)| \leq \mu|z|, \quad z \in r\mathbb{B}^q, n \geq 0.$$

and

$$\mu^h < \nu. \quad (2.4)$$

The *abstract basin of attraction* or *tail space*  $\Omega$  of the family  $(\varphi_{u_n, u_{n+1}} : r\mathbb{B}^q \rightarrow r\mathbb{B}^q)$  is defined in [11] (see also [1]) as its topological inductive limit endowed with a natural complex structure.  $\Omega$  is the quotient of the set

$$\left\{ z \in \prod_{m \geq n} r\mathbb{B}^q : n \in \mathbb{N}, z_{m+1} = \varphi_{u_m, u_{m+1}}(z_m), \quad 0 \leq n \leq m \right\},$$

obtained identifying  $z$  and  $z'$  if  $z_m = z'_m$  for  $m$  large enough, and the holomorphic structure is induced by a family of open inclusions  $(\omega_n : r\mathbb{B}^q \rightarrow \Omega)$  defined as

$$\omega_n(z) \doteq (\varphi_{u_n, u_m}(z))_{m \geq n}, \quad n \in \mathbb{N},$$

which are thus by definition biholomorphisms with their image and satisfy

$$\omega_n(z) = \omega_m \circ \varphi_{u_n, u_m}(z), \quad 0 \leq n \leq m, z \in r\mathbb{B}^q. \quad (2.5)$$

By [11, Theorem 2.2] there exists an univalent mapping  $\Psi : \Omega \rightarrow \mathbb{C}^q$ . We claim that, for all  $s \geq 0$ , the sequence  $(\Psi \circ \omega_m \circ \varphi_{s, u_m})_{m \geq 0}$  converges uniformly on compact sets in  $\text{Hol}(\mathbb{B}^q, \mathbb{C}^q)$ . Indeed by equation (2.2) and assumption a) one has that for all  $s \geq 0$ ,

$$\lim_{m \rightarrow \infty} \varphi_{s, u_m}(z) = 0,$$

uniformly on compact sets. Thus, if  $0 < v < 1$ , there exist  $m(v) \in \mathbb{N}$  such that for all  $j \geq m(v)$ , one has  $\varphi_{s, u_j}(v\mathbb{B}^q) \subset r\mathbb{B}^q$ . Let  $j, h$  be integers such that  $m(v) \leq j \leq h$ , then by (2.5),

$$\Psi \circ \omega_h \circ \varphi_{s, u_h}(z) = \Psi \circ \omega_j \circ \varphi_{s, u_j}(z), \quad z \in v\mathbb{B}^q.$$

Thus the sequence  $(\Psi \circ \omega_m \circ \varphi_{s, u_m})$  is eventually constant in  $\text{Hol}(v\mathbb{B}^q, \mathbb{C}^q)$ .

Let  $f_t : \mathbb{B}^q \rightarrow \mathbb{C}^q$  the univalent mapping defined as

$$f_t(z) \doteq \lim_{m \rightarrow +\infty} \Psi \circ \omega_m \circ \varphi_{t, u_m}(z). \quad (2.6)$$

One easily verifies that

$$f_s(z) = f_t \circ \varphi_{s,t}(z), \quad 0 \leq s \leq t, z \in \mathbb{B}^q, \quad (2.7)$$

and that

$$\bigcup_{t \geq 0} f_t(\mathbb{B}^q) = \Psi(\Omega).$$

Notice that the abstract basin of attraction of the family  $(\varphi_{u_n, u_{n+1}})$  is thus biholomorphic to the Loewner range of the family  $(\varphi_{s,t})$  defined in [5]. This can be checked directly since both objects are defined as direct limits.

By [5, Theorem 4.10] one has that  $(f_t: \mathbb{B}^q \rightarrow \mathbb{C}^q)$  is a Loewner chain of order  $\infty$ , that is

- $f_s(\mathbb{B}^q) \subset f_t(\mathbb{B}^q)$  for all  $0 \leq s \leq t$ ,
- for any compact set  $K \subset \mathbb{B}^q$  and for any  $T > 0$  there exists a  $k_{K,T} > 0$  satisfying

$$|f_t(z) - f_s(z)| \leq k_{K,T}(t - s), \quad z \in K, \quad 0 \leq s \leq t < T. \quad (2.8)$$

By [5, Theorem 5.2] one obtains finally

$$\frac{\partial f_t(z)}{\partial t} = Df_t(z)h(z, t), \quad z \in \mathbb{B}^q, \text{ a.e. } t \geq 0.$$

Thus any univalent mapping  $\Psi: \Omega \rightarrow \mathbb{C}^q$  gives rise to a solution  $(f_t: \mathbb{B}^q \rightarrow \mathbb{C}^q)$  of the Loewner PDE. Following [1, Remark A.4] we recall a way to construct such univalent mapping  $\Psi: \Omega \rightarrow \mathbb{C}^q$ . Given any polynomial map  $p: \mathbb{C}^q \rightarrow \mathbb{C}^q$  of degree at most  $k$  there exists [13][29] an holomorphic automorphism  $\Phi$  of  $\mathbb{C}^q$  such that

$$\Phi(z) = p(z) + O(|z|^{k+1}).$$

We choose a sequence of automorphisms  $(\Phi_{n,n+1}: \mathbb{C}^q \rightarrow \mathbb{C}^q)$  satisfying

$$\Phi_{n,n+1}(z) = \varphi_{u_n, u_{n+1}}(z) + O(|z|^h), \quad n \geq 0,$$

where  $h \in \mathbb{N}$  is as in (2.4). We denote the basin of attraction of the sequence  $(\Phi_{n,n+1})$  by

$$\mathfrak{A}(\Phi_{n,n+1}) \doteq \{z \in \mathbb{C}^q : \lim_{n \rightarrow \infty} \Phi_{n-1,n} \circ \cdots \circ \Phi_{0,1} = 0\}.$$

It follows from [1, Remark A.4] that there exists a biholomorphism

$$\Psi: \Omega \rightarrow \mathfrak{A}(\Phi_{n,n+1}) \subset \mathbb{C}^q.$$

If  $\ell < 2$ , then  $h = 2$  and, by [30, Theorem 4], one has that the basin of attraction  $\mathfrak{A}(\Phi_{n,n+1})$  is biholomorphic to  $\mathbb{C}^q$ .

By [5, Theorem 4.10] any solution  $(g_t: \mathbb{B}^q \rightarrow \mathbb{C}^q)$  of the Loewner PDE has to satisfy  $g_s = g_t \circ \varphi_{s,t}$  for all  $0 \leq s \leq t$  and thus [5, Theorem 4.7] yields that the family  $(g_t)$  is of the form  $(\Lambda \circ f_t)$ , where  $\Lambda: \bigcup_{t \geq 0} f_t(\mathbb{B}^q) \rightarrow \mathbb{C}^q$  is holomorphic.

□

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